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Anomalous Fluctuation and Relaxation in Unstable Systems

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The essential ideas of the scaling theory of transient phenomena proposed by the author for a single macrovariable near the instability point are extended to multi-macrovariables in nonequilibrium systems. The time region is divided into three regimes according to the scaling behavior of the fluctuating parts of the macrovariables. In the first regime, the fluctuation is Gaussian and it is described by the linearized stochastic equation (or linear Fokker–Planck equation). In the second regime, the fluctuation is non-Gaussian, but it is probabilistic or stochastic (not dynamical) in the sense that the stochastic nature comes from the probability distribution in the initial regime and that each representative motion is deterministic, namely a random force can be neglected asymptotically in the second regime. In the final regime, the fluctuation is again Gaussian. A fluctuation-enhancement theorem for multi-macrovariables is given, which states that the fluctuation becomes enhanced by the order of the system size Ω in the second regime, which is of order $\log \Omega$, if the initial system is located just at the unstable point. An anomalous fluctuation theorem for multi-macrovariables is also proven, which states that the fluctuation is anomalously enhanced in proportion to δ^{-2} at times of order $\log \delta$ if the initial system deviates by δ from the unstable point.

KEY WORDS: Macrovariable; multi-macrovariable; multimode; most probable path; variance; instability point; unstable system; fluctuation enhancement; anomalous fluctuation; relaxation; mode coupling; scaling property; scaling theory; Gaussian; non-Gaussian; linear, initial regime; nonlinear, second regime; nonequilibrium system; asymptotic evaluation; Ω-expansion; Fokker–Planck equation; Kramers–Moyal equation.

1. INTRODUCTION

In a series of papers,[1–4] a general theory on relaxation and fluctuation near the instability point has been proposed by the present author and has been

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applied to the laser model[1,4,5] and to superradiance[6] near the unstable point.[2] The previous arguments have concentrated on a single macrovariable for the sake of simplicity and mathematical rigor. However, the essence of the previous papers on a single macrovariable can be easily extended to multi-macrovariables (or many modes), as was announced in Ref. 1. The purpose of this paper is to give explicitly a general scheme for the extension of the scaling theory[1-4] for a single macrovariable to multicomponent systems.

In Section 2, the essence of the scaling theory for a single macrovariable is rephrased in order to show how it can be extended to multi-macrovariables or an infinite number of macrovariables. In Section 3, a general scheme for the extension to multicomponent systems is given and a *fluctuation-enhancement theorem* for multi-macrovariables is derived. In Section 4, the anomalous fluctuation effect is also proven to occur in multicomponent systems, quite similar to the case of a single mode,[2,10] namely the fluctuation is anomalously enhanced inversely proportional to the square of the deviation δ of the initial system from the unstable equilibrium point.

2. ESSENCE OF THE SCALING THEORY OF A SINGLE MACROVARIABLE

It will be very instructive to review the essence of the scaling theory[1-4] of a single macrovariable near the instability point. The time region is divided into three regimes as shown in Fig. 1, according to the scaling behavior of the fluctuating parts of the macrovariables, namely (i) linear, Gaussian, initial regime; (ii) nonlinear, non-Gaussian, anomalous fluctuation, drift or scaling regime; (iii) linear, Gaussian, final regime.

If $x(t)$ is an intensive macrovariable defined by

$$x(t) = X(t)/\Omega; \qquad \Omega = \text{system size} \tag{1}$$

we can separate[11] it into two parts as

$$x(t) = y(t) + z(t) \tag{2}$$

where $y(t)$ denotes the most probable path of $x(t)$, and $z(t)$ is the remaining

[2] For related work on unstable systems, see Refs. 7–9.

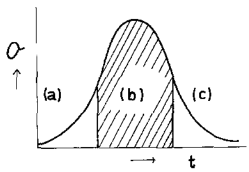

Fig. 1. σ: Fluctuation; (a) initial regime, (b) second, nonlinear regime, (c) final regime.

fluctuating part. Then, the above three regimes are classified as follows:

$$\begin{array}{lll} \text{(i)} & z(t) = O(\Omega^{-1/2}) & \text{in the initial regime} \\ \text{(ii)} & z(t) = O(1) & \text{in the second regime} \\ \text{(iii)} & z(t) = O(\Omega^{-1/2}) & \text{in the final regime} \end{array} \tag{3}$$

That is, the fluctuating part $z(t)$ differs in scaling behavior with respect to the system size in each regime, when the system starts from (or near) the unstable point. This will be the simplest criterion for the classification of the above three regimes.

In the *initial regime*, the intrinsic fluctuation is very small (of order $\Omega^{-1/2}$) for a large Ω, and consequently a random force acting on the system plays an essential role in this regime. Otherwise, the system does not change its state under the unstable equilibrium initial condition. Furthermore, the nonlinearity of the system is not important in this regime, because the deviation of $x(t)$ from the unstable point x_0 is small. Therefore, the temporal evolution of $x(t)$ in this initial regime is *Gaussian* and its distribution function satisfies the linear Fokker–Planck equation:

$$\frac{\partial}{\partial t} P(x, t) = \left\{ -\frac{\partial}{\partial x} \gamma(x - x_0) + \frac{1}{2} c\epsilon \frac{\partial^2}{\partial x^2} \right\} P(x, t) \tag{4}$$

where $\epsilon = \Omega^{-1}$, $\gamma = c_1'(x_0)$, and $c = c_2(x_0)$, $c_n(x)$ being the nth moment of the transition probability. The solution of (4) with an initial condition $P_0(x)$ at $t = 0$ is given by

$$P_{\text{ini}}(x, t) = \frac{1}{[2\pi\epsilon\sigma_2(t)]^{1/2}} \int_{-\infty}^{\infty} P_0(y) \exp\left\{ -\frac{[x - y(t)]^2}{2\epsilon\sigma_2(t)} \right\} dy \tag{5}$$

where

$$y(t) = (y - x_0)e^{\gamma t} + x_0; \qquad \sigma_2(t) = \sigma_1(e^{2\gamma t} - 1); \qquad \sigma_1 = c/(2\gamma) \tag{6}$$

In particular, if $P_0(x)$ is Gaussian, namely

$$P_0(x) = \frac{1}{(2\pi\epsilon\sigma_0)^{1/2}} \exp - \frac{(x - x_0)^2}{2\epsilon\sigma_0} \tag{7}$$

then $P(x, t)$ takes the following Gaussian form:

$$P_{\text{ini}}(x, t) = \frac{1}{[2\pi\epsilon\sigma(t)]^{1/2}} \exp - \frac{(x - x_0)^2}{2\epsilon\sigma(t)} \tag{8}$$

where the variance $\sigma(t)$ is now given by

$$\sigma(t) = (\sigma_0 + \sigma_1)e^{2\gamma t} - \sigma_1 \tag{9}$$

as is well known. The fluctuating part $z(t)$ in (8) is given by

$$z(t) \sim [\epsilon\sigma(t)]^{1/2}, \qquad \text{or} \qquad \sigma(t) \sim \epsilon^{-1}z^2(t) \tag{10}$$

and it becomes larger and larger as the time t increases, although it is of order $\Omega^{-1/2}$ for a small t. When t increases to the order

$$t \sim t_1 \equiv (2\gamma)^{-1} \log[\Omega/(\sigma_0 + \sigma_1)] \tag{11}$$

the fluctuating part $z(t)$ becomes of order unity [i.e., $z(t) \sim O(1)$], and thus the system is in the second, non-Gaussian regime for $t \sim t_1$. Namely, the linear, Gaussian approximation (4) breaks down in the time region (11). However, our Gaussian approximation[11] predicts qualitatively what happens in the second regime. In fact, it shows that *the fluctuation (or variance) is anomalously enhanced up to order unity* (*or* $\Omega \equiv \epsilon^{-1}$) *in the second regime.* Note that this anomalous fluctuation of order unity is maximum from the definition of the fluctuation [cf. the fluctuation $\langle x^2\rangle_c = \langle x^2\rangle - \langle x\rangle^2 \leqslant \langle x^2\rangle \leqslant$ (maximum value of $x)^2 \sim O(1)$]. Thus, the scaling behavior of $z(t)$ in the second regime is concluded to be $z(t) = O(1)$. This will be used effectively in the non-Gaussian, scaling regime.

In the *second regime* (or *scaling regime*), the fluctuating part $z(t)$ becomes of order unity, namely it satisfies the scaling property

$$z(t) \sim O(1) \tag{12}$$

(i.e., invariant for scaling of the size). This indicates that *the distribution function is very broad and consequently that the diffusion effect is neglected effectively for a large system size* Ω. Therefore, the distribution of x is governed asymptotically by the drift equation in the second regime. More explicitly, we discuss the Kramers–Moyal equation[1–4,10–12]

$$\epsilon \frac{\partial}{\partial t} P(x, t) = -\mathscr{H}\left(x, \epsilon \frac{\partial}{\partial x}\right) P(x, t) \tag{13}$$

where

$$\mathscr{H}(x, p) = \int (1 - e^{-rp}) w(x, r)\, dr = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} p^n c_n(x) \tag{14}$$

and

$$c_n(x) = \int r^n w(x, r)\, dr \tag{15}$$

with the transition probability $w(x, r)$. Assuming that all $c_n(x)$ are of order unity as usual, the scaling property of the nth term of the right-hand side of (13) is of order Ω^{-n+1} in the second regime, because all derivatives of $P(x, t)$ with respect to x are of order unity from the scaling property (12). Thus, the first drift term is dominant in the second regime. That is, $P(x, t)$ satisfies asymptotically the following *drift equation*:

$$\frac{\partial}{\partial t} P(x, t) + \frac{\partial}{\partial x} c_1(x) P(x, t) = 0 \tag{16}$$

This has been proven more rigorously in previous papers[1-4] by using the generalized scaling transformation of time of the form

$$\tau = \epsilon e^{2\gamma t} \tag{17}$$

The characteristic equations of the partial differential equation (16) are

$$dx/dt = c_1(x) \tag{18}$$

and

$$dP/dt = -c_1'(x)P \tag{19}$$

Equation (18) corresponds to the temporal evolution equation of the most probable path $y(t)$. That is, all the phase points of $P(x, t)$ are governed by the classical (or deterministic) equation (18), as shown in Fig. 2. From (19), the solution of (16) with the initial condition $P(x, t_1) = P_{\text{ini}}(x, t_1)$ at $t = t_1$ is given by

$$P(x, t) = \{c_1(x_1(x, t))\}\{c_1(x)\}^{-1}P_{\text{ini}}(x_1(x, t), t_1) \tag{20}$$

where $x_1(x, t)$ is the inverse function of the solution of (18) with the initial condition $x = x_1$ at $t = t_1$: $x = x(x_1, t)$. If we define a function $f(x)$ by[2]

$$f(x) \equiv \exp\left\{\gamma \int_{a_0}^{x} [c_1(\xi)]^{-1}\, d\xi\right\} \tag{21}$$

a_0 being determined so that $f'(x_0) = 1$, then it is easily shown that the first factor of (20), $c_1(x_1)/c_1(x)$, is the Jacobian of the transformation $x_1 \to x$:

$$\frac{\partial x_1}{\partial x} = \frac{f'(x)}{f'(x_1)} e^{-\gamma t} = \frac{c_1(x_1)}{c_1(x)} \tag{22}$$

Thus, the transformation $x_1 \to x$ [or the evolution (16)] conserves the probability.[13] In particular, if we use the Gaussian solution (8) in the initial regime, we obtain[1-4]

$$P(x, t) = \frac{1}{(2\pi\tau)^{1/2}} \frac{f'(x)}{f'(x_1(x, t))} \times \exp\left(-\frac{\exp(-2\gamma t_1)}{2\epsilon\sigma}\{f^{-1}(f(x)\exp[-\gamma(t - t_1)])\}^2\right) \tag{23}$$

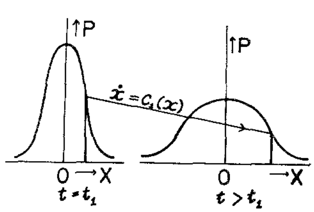

Fig. 2. Schematic time dependence of the distribution function in the second regime.

where

$$\tau = (\sigma_0 + \sigma_1)\epsilon e^{2\gamma t} - \epsilon\sigma_1 \cong \sigma\epsilon e^{2\gamma t}; \qquad \sigma \equiv \sigma_0 + \sigma_1 \tag{24}$$

This is reduced asymptotically to the following scaling form:

$$P_{\rm sc}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} f'(x) \exp - \frac{f^2(x)}{2\tau} \tag{25}$$

in the second nonlinear regime. It should be remarked that the distribution function depends upon the inverse system size ϵ only through the scaling time variable τ in the second, nonlinear regime.

In the *final regime*, we may expect a normal Gaussian fluctuation for $z(t)$. Then, the distribution function satisfies[3] asymptotically the following linear Fokker–Planck equation:

$$\frac{\partial}{\partial t} P_{\rm fin} = \left\{\frac{\partial}{\partial x}[\gamma_e(x - x_e)] + \frac{\hat{c}\epsilon}{2}\frac{\partial^2}{\partial x^2}\right\} P_{\rm fin} \tag{26}$$

where x_e is the stable equilibrium point,

$$\gamma_e = -c_1'(x_e) > 0, \qquad \text{and} \qquad \hat{c} = c_2(x_e) \tag{27}$$

The solution of (26) with the initial condition $P_{\rm fin}(x, t_2) = P_{\rm sc}(x, \tau(t_2))$ at a time t_2 in the boundary region between the scaling and final regimes is expressed by[3]

$$P_{\rm fin}(x, t) = \frac{1}{[2\pi\epsilon\sigma_f(t)]^{1/2}} \int_{-\infty}^{\infty} P_{\rm sc}(y, \tau(t_2)) \exp - \frac{[x - y(t)]^2}{2\epsilon\sigma_f(t)}\, dy \tag{28}$$

where

$$y(t) = (y - x_e)e^{-\gamma_e(t-t_2)} + x_e \tag{29a}$$

and

$$\sigma_f(t) = \sigma_f\{1 - \exp[-2\gamma_e(t - t_2)]\}; \qquad \sigma_f = \hat{c}(2\gamma_e)^{-1} \tag{29b}$$

This approaches the correct equilibrium state $P_{\rm eq}(x)$ given by

$$P_{\rm eq}(x) \propto \exp[-(x - x_e)^2(2\epsilon\sigma_f)^{-1}] \tag{30}$$

near the stable equilibrium point x_e. The qualitative behavior of $P_{\rm fin}(x, t)$ is independent of a choice of t_2. In the case where there exist several stable equilibrium points, we divide the x region into subregions each of which contains only one stable equilibrium point inside and we define an initial function $P_{\rm sc}^{(j)}(x, \tau(t_2))$ in the jth subregion, which is equal to $P_{\rm sc}(x, \tau(t_2))$ in the corresponding subregion and vanishes outside. We can repeat the above procedure to construct $P_{\rm fin}^{(j)}(x, t)$ for each initial function $P_{\rm fin}^{(j)}(x, \tau(t_2))$. The required distribution function $P_{\rm fin}^{(j)}(x, t)$ in the final regime is given by the sum of $\{P_{\rm fin}^{(j)}(x, t)\}$. For the symmetric case of two stable equilibrium points

$\pm x_e$ such as the laser model discussed in the previous papers,[1-5] the solution in the final regime is given by

$$P_{\mathrm{fin}}(x, t) = \frac{1}{[2\pi\epsilon\sigma_f(t)]^{1/2}} \left\{ \int_0^{\infty} P_{\mathrm{sc}}^{(1)}(y, \tau(t_2)) \exp - \frac{[x - y_+(t)]^2}{2\epsilon\sigma_f(t)}\, dy + \int_{-\infty}^{0} P_{\mathrm{sc}}^{(2)}(y, \tau(t_2)) \exp - \frac{[x - y_-(t)]^2}{2\epsilon\sigma_f(t)}\, dy \right\} \tag{31a}$$

where $y_\pm(t) = (y \mp x_e)\exp[-\gamma_e(t - t_2)] \pm x_e$, and $P_{\mathrm{sc}}^{(1)}$ corresponds to the scaling solution in the positive x region and $P_{\mathrm{sc}}^{(2)}$ to that in the negative x region. Clearly, the above solution approaches asymptotically the correct equilibrium state with two Gaussian peaks around $x = \pm x_e$. An alternative connection which is essentially equivalent to (31a) is given by

$$P_{\mathrm{fin}}(x, t) = \frac{1}{2[2\pi\epsilon\sigma_f(t)]^{1/2}} \int_{-\infty}^{\infty} P_{\mathrm{sc}}(y, \tau(t_2)) \times \left[\exp - \frac{[x - y_+(t)]^2}{2\epsilon\sigma_f(t)} + \exp - \frac{[x - y_-(t)]^2}{2\epsilon\sigma_f(t)} \right] dy \tag{31b}$$

An explicit result for the laser model[1-5] is shown in Appendix A.

It also will be useful to explain qualitatively the anomalous fluctuation effect[2,10] when the initial system deviates from the unstable point by δ; i.e., $y_0 = x_0 + \delta$ ($\delta \ll 1$), in the *extensive region* ($\epsilon^\mu \ll \delta$, with an appropriate positive exponent μ, which is equal to 1/2 in a normal situation), as shown in Fig. 3. The essence of the anomalous fluctuation theorem proven in a previous paper[2] is reinterpreted as follows. In the extensive region, the fluctuating part $z(t)$ is Gaussian, and consequently, following van Kampen[11] and Kubo *et al.*,[10,12,15] the temporal evolution equations of the most probable path $y(t)$ and variance $\sigma(t)$ are given by

$$\frac{d}{dt} y(t) = c_1(y(t)), \qquad \frac{d}{dt}\sigma(t) = 2c_1'(y)\sigma(t) + c_2(y) \tag{32}$$

respectively. Although these equations are easily integrated,[10,12] we discuss here the qualitative features by linearizing Eq. (32) around the unstable point x_0 as follows:

$$\frac{d}{dt}(\delta y) = \gamma(\delta y); \qquad \delta y = y(t) - x_0, \quad \gamma = c_1'(x_0) > 0 \tag{33}$$

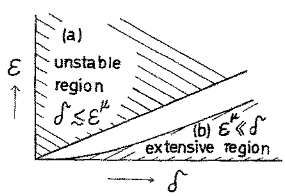

Fig. 3. ϵ–δ plane; (a) unstable regime $\delta \lesssim \epsilon^\mu$; (b) extensive regime $\epsilon^\mu \ll \delta$; where $\mu = 1/2$ in an ordinary situation.

and

$$\frac{d}{dt}\sigma_l(t) = 2\gamma\sigma_l(t) + c_2(x_0) \tag{34}$$

where we have used the condition that $c_1(x_0) = 0$. The solution of (33) is

$$\delta y = \delta e^{\gamma t}; \qquad \delta \equiv (\delta y)_{t=0} = y_0 - x_0 \tag{35}$$

The above linear approximation is valid only when

$$|\delta y| \lesssim \Delta \tag{36}$$

with a certain constant Δ of order 1 (or $\Delta \ll 1$).

For the time region $\delta y \simeq \Delta$, namely for

$$t \sim t_1 \sim \gamma^{-1}\log(\Delta/\delta) \tag{37}$$

the linear variance $\sigma_l(t)$, which is the solution of (34), becomes very large or anomalously enhanced as

$$\sigma_l(t_1) = (\sigma_0 + \sigma_1)e^{2\gamma t_1} - \sigma_1 \simeq (\sigma_0 + \sigma_1)\frac{\Delta^2}{\delta^2} \sim \frac{1}{\delta^2} \tag{38a}$$

with $\sigma_1 = c_2(0)/(2\gamma)$, if $(\sigma_0 + \sigma_1)$ is nonvanishing. The saturation comes from the nonlinear effect neglected in the above linear approximation. This gives an intuitive explanation of the previous anomalous fluctuation theorem[(2)]:

$$\sigma(t, \delta) \simeq \sigma_{sc} = \frac{\sigma_0 + \sigma_1}{\delta^2}\left[\frac{c_1(y_{sc}(\tau))}{\gamma}\right]^2 \tag{38b}$$

where $y_{sc}(\tau)$ is the scaling solution of the first equation in (32). That is, there occurs an anomalous fluctuation inversely proportional to the square of the

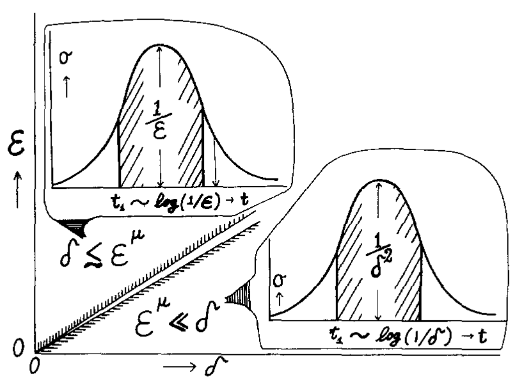

Fig. 4. Qualitative features of the anomalous fluctuation in the extensive region and fluctuation enhancement in the unstable region.

deviation δ of the initial system from the unstable point x_0 in the time region proportional to $\log(1/\delta)$. The above situation is illustrated in Fig. 4, together with the fluctuation enhancement effect in the unstable regime. This treatment will be extended to multi-macrovariables in Section 4.

An alternative formulation of the scaling theory will be given in Appendix B, which is applicable both to the unstable region and to the extensive region. This formulation is based on the following Ansatz:

$$P(x, t) \sim C \exp[\epsilon^{-1}\varphi_0(x, t) + \varphi_1(x, t)] \tag{39}$$

for small ϵ. Here it should be remarked that the *second term* $\varphi_1(x, t)$ *is of the same order as the first extensive term. Namely, both terms are equally important in the second scaling regime.*

3. EXTENSION TO MULTIMODES

The essence of the scaling theory for a single macrovariable presented in the previous section can be easily extended to multicomponent systems. For simplicity, we consider here the following Kramers–Moyal equation for multi-macrovariables[10,12,14] $\{X_j\}$:

$$\epsilon \frac{\partial}{\partial t} P(\mathbf{x}, t) + \mathscr{H}\left(\mathbf{x}, \epsilon \frac{\partial}{\partial \mathbf{x}}\right) P(\mathbf{x}, t) = 0 \tag{40}$$

with the use of the vector notation defined by

$$\mathbf{x} = \mathbf{X}/\Omega; \qquad \mathbf{X} = (X_1, X_2, X_3, \ldots, X_n) \tag{41}$$

where

$$\mathscr{H}(\mathbf{x}, \mathbf{p}) = \int [1 - \exp(\mathbf{r}\cdot\mathbf{p})] w(\mathbf{x}, \mathbf{r}, t)\, d\mathbf{r} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} \mathbf{p}^m c_m(\mathbf{x}, t) \tag{42}$$

and $c_m(\mathbf{x}, t)$ is a tensor of degree m defined by

$$c_m(\mathbf{x}, t) = \int \mathbf{r}^m w(\mathbf{x}, \mathbf{r})\, d\mathbf{r} \tag{43}$$

More generally, we may consider a wavenumber-dependent variable x_k.

The essential point of our extension is to divide the time region into the following three regimes, according to the scaling behavior of a fluctuating part $\mathbf{z}(t)$ [or $z_k(t)$] defined by

$$\mathbf{x}(t) = \mathbf{y}(t) + \mathbf{z}(t) \qquad \text{or} \qquad x_k(t) = y_k(t) + z_k(t) \tag{44}$$

(i) Linear, Gaussian, initial regime, in which

$$\mathbf{z}(t) = O(\Omega^{-1/2}) \qquad \text{or} \qquad z_k \to z'_{k'} = b^{-\alpha} z_k \tag{45}$$

(ii) Nonlinear, non-Gaussian, second regime, in which

$$\mathbf{z}(t) = O(1) \qquad \text{or} \qquad z_k \to z'_{k'} = z_k \tag{46}$$

(iii) Linear, Gaussian, final regime, in which

$$\mathbf{z}(t) = O(\Omega^{-1/2}) \qquad \text{or} \qquad z_k \to z'_{k'} = b^{-\alpha} z_k \tag{47}$$

for the scale transformation

$$L \text{ (length)} \to L' = bL, \qquad k \to k' = b^{-1}k,$$
$$\Omega \equiv L^d \to \Omega' = b^d \Omega \tag{48}$$

Here, the scaling exponent α takes the value $\frac{1}{2}d$ in usual situations corresponding to the Gaussian fluctuation of order $\Omega^{-1/2}$ (where d denotes the dimensionality of the system), but it is, in general, permitted to assume an arbitrary positive value.

From the above scaling properties of $\mathbf{z}(t)$ [or $z_k(t)$], the temporal evolution equation of $P(\mathbf{x}, t)$ can be reduced[3] to (i) a linear Fokker–Planck equation in the initial regime

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial \mathbf{x}} \cdot \boldsymbol{\gamma} \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{\epsilon}{2} \frac{\partial}{\partial \mathbf{x}} \mathbf{D} \frac{\partial}{\partial \mathbf{x}} \right] P(\mathbf{x}, t) \tag{49}$$

(ii) a nonlinear drift equation in the second regime

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial \mathbf{x}} c_1(\mathbf{x}) P(\mathbf{x}, t) = 0 \tag{50}$$

(iii) a linear Fokker–Planck equation in the final regime

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial \mathbf{x}} \boldsymbol{\gamma}_e (\mathbf{x} - \mathbf{x}_e) + \frac{\epsilon}{2} \frac{\partial}{\partial \mathbf{x}} \mathbf{D}_e \frac{\partial}{\partial \mathbf{x}} \right] P \tag{51}$$

if there exists a stable equilibrium point $\mathbf{x}_e$. Here $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}_e$, $\mathbf{D}$, and $\mathbf{D}_e$ are constant matrices. That is, the fluctuating random-force effect of the diffusion term (and higher terms) in (40) can be neglected effectively in the second regime, according to the scaling behavior (46). Note that the order of the nth term in (40) is Ω^{-n+1}, as in the case of a single mode. The above scheme is explained graphically in Fig. 5.

The next important point is how to evaluate the order of the fluctuating part $\mathbf{z}(t)$ and how to estimate the magnitude of each time regime or boun-

[3] The main idea in the present paper was reported at a joint meeting on plasma and nonequilibrium statistical mechanics organized by R. Kubo and held at the Institute of Plasma Physics, Nagoya University, March 15, 1976.

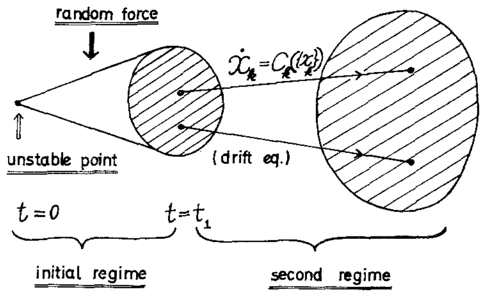

Fig. 5. Schematic explanation of the temporal evolution of the system in the initial and second regimes.

daries of the three time regimes. For this purpose, we start from Eq. (40) linearized around the unstable equilibrium point $\mathbf{x}_0$:

$$\frac{\partial}{\partial t} P = \left[-\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{K}_0 \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{\epsilon}{2} \frac{\partial}{\partial \mathbf{x}} \mathbf{D}_0 \frac{\partial}{\partial \mathbf{x}} \right] P(\mathbf{x}, t) \tag{52}$$

where we have approximated (40) up to its second derivative. Here, the matrices $\mathbf{K}_0$ and $\mathbf{D}_0$ are defined by

$$\mathbf{K}_0 = \mathbf{c}_1'(\mathbf{x}_0) \equiv \frac{\partial}{\partial \mathbf{x}_0} \mathbf{c}_1(\mathbf{x}_0), \qquad \mathbf{D}_0 = (c_2^{ij}(\mathbf{x}_0)) \tag{53}$$

The solution of (52) with the initial Gaussian distribution of a variance $\boldsymbol{\sigma}(0)$ is given by

$$P(\mathbf{x}, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \frac{1}{[\det \boldsymbol{\sigma}(t)]^{1/2}} \exp\left\{ -(\mathbf{x} - \mathbf{x}_0) \left(\frac{1}{2\epsilon \boldsymbol{\sigma}(t)} \right) (\mathbf{x} - \mathbf{x}_0) \right\} \tag{54}$$

Here the variance matrix $\boldsymbol{\sigma}(t)$ satisfies the equation[10,14]

$$\frac{d}{dt} \boldsymbol{\sigma}(t) = \mathbf{K}_0 \boldsymbol{\sigma}(t) + \boldsymbol{\sigma}(t) \tilde{\mathbf{K}}_0 + \mathbf{D}_0 \tag{55}$$

where $\tilde{\mathbf{K}}_0$ denotes the transposed matrix of $\tilde{\mathbf{K}}_0$. The formal solution of (55) is expressed by[16,17]

$$\boldsymbol{\sigma}(t) = \exp(\mathbf{K}_0 t)\, \boldsymbol{\sigma}(0) \exp(\tilde{\mathbf{K}}_0 t) + \int_0^t \exp[\mathbf{K}_0(t - t')]\, \mathbf{D}_0 \exp[\tilde{\mathbf{K}}_0(t - t')]\, dt' \tag{56}$$

We have an alternative, simpler expression of the form

$$\boldsymbol{\sigma}(t) = [\exp(\mathbf{K}_0 t)][\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_0] \exp(\tilde{\mathbf{K}}_0 t) + \boldsymbol{\sigma}_0 \tag{57}$$

when there exists the solution $\boldsymbol{\sigma}_0$ of the algebraic equation

$$\mathbf{K}_0 \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_0 \tilde{\mathbf{K}}_0 + \mathbf{D}_0 = 0 \tag{58}$$

In order to study the scaling behavior of $\mathbf{z}(t)$ and to find when the system enters the second regime, we first investigate in detail the temporal evolution of the variance of fluctuation $\epsilon\boldsymbol{\sigma}(t) \simeq \langle(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)\rangle$ in the above linear approximation. If $\mathbf{K}_0$ is semisimple (i.e., diagonalizable), and $\{\gamma_j\}$ are its eigenvalues, then the relaxation modes of $\boldsymbol{\sigma}(t)$ are given by[16] a set of $\exp[(\gamma_i + \gamma_j)t]$, namely

$$\sigma_{kl}(t) = \sum_{i,j} a_{ij}^{kl} \exp[(\gamma_i + \gamma_j)t] + b_{kl} \tag{59}$$

where $\{a_{ij}^{kl}\}$ and b_{kl} are appropriate constants. For a more explicit expression for $\boldsymbol{\sigma}(t)$, see the review article by van Kampen[16] for a diagonalizable $\mathbf{K}_0$. This result shows how fast the fluctuation grows or decays.

In general, an arbitrary matrix $\mathbf{K}_0$ is expressed by a sum of a semisimple matrix $\mathbf{S}$ and a nilpotent matrix $\mathbf{N}$ as

$$\mathbf{K}_0 = \mathbf{S} + \mathbf{N}; \qquad \mathbf{SN} = \mathbf{NS} \tag{60}$$

Let eigenvalues of $\mathbf{S}$ (and equivalently those of $\mathbf{K}_0$) be $\{\gamma_j\}$. Then, $\mathbf{K}_0$ takes the following Jordan normal form after an appropriate regular transformation $\mathbf{P}$:

$$\mathbf{P}\mathbf{K}_0\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{K}_1 & & & 0 \\ & \mathbf{K}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{K}_s \end{bmatrix}; \qquad \mathbf{K}_j = \begin{bmatrix} \gamma_j & 1 & & 0 \\ & \gamma_j & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & \gamma_j \end{bmatrix} \tag{61}$$

Each submatrix $\mathbf{K}_j$ is expressed by a sum of a subsemisimple matrix $\mathbf{S}_j$ and a subnilpotent matrix $\mathbf{N}_j$ defined by

$$\mathbf{S}_j = \begin{bmatrix} \gamma_j & & & 0 \\ & \gamma_j & & \\ & & \ddots & \\ 0 & & & \gamma_j \end{bmatrix}, \qquad \mathbf{N}_j = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & 0 \end{bmatrix} \tag{62}$$

in n_j dimensions. The regular transformation $\mathbf{P}$ transforms $\boldsymbol{\sigma}(t)$ into the following:

$$\begin{aligned}\boldsymbol{\sigma}'(t) = \mathbf{P}\sigma(t)\tilde{\mathbf{P}} &= \exp(\mathbf{K}_0't)\,\boldsymbol{\sigma}'(0)\exp(\tilde{\mathbf{K}}_0't) \\ &+ \int_0^t \exp[\mathbf{K}_0'(t-t')]\,\mathbf{D}_0'\exp[\tilde{\mathbf{K}}_0'(t-t')]\,dt'\end{aligned} \tag{63}$$

where

$$\mathbf{K}_0' = \mathbf{P}\mathbf{K}_0\mathbf{P}^{-1}, \qquad \boldsymbol{\sigma}'(0) = \mathbf{P}\boldsymbol{\sigma}(0)\tilde{\mathbf{P}}, \qquad \mathbf{D}_0' = \mathbf{P}\mathbf{D}_0\tilde{\mathbf{P}} \tag{64}$$

Therefore, a representative time dependence of the variance $\sigma'_{ij}(t)$ corresponding to the ith and jth modes in the above representation is given by

$$\begin{aligned}\exp(t\mathbf{K}_i)\,\sigma'_{ij}(0)\exp(t\tilde{\mathbf{K}}_j) &= [\exp(t\mathbf{S}_i)][\exp(t\mathbf{N}_i)\,\sigma'_{ij}(0)\exp(t\mathbf{N}_j)]\exp(tS_j)\\ &= \exp[t(\gamma_i + \gamma_j)]\cdot\sigma'_{ij}(0, t)\end{aligned} \tag{65}$$

where $\sigma'_{ij}(0, t)$ is a submatrix of the form $n_i \times n_j$ whose matrix elements are all polynomials of t, at most, of the order of $(n_i + n_j - 2)$. Thus, we obtain the following theorem:

Theorem 1. For an arbitrary time-independent regression matrix $\mathbf{K}_0$ whose eigenvalues are $\{\gamma_j\}$, the variance $\sigma'_{ij}(t)$ corresponding to the modes γ_i and γ_j takes the following exponential growth or decay with extra factors of polynomials of t:

$$\sigma'_{ij}(t) = \hat{\sigma}_{ij}(0, t)\exp[t(\gamma_i + \gamma_j)] + \text{const} \tag{66}$$

and the order of the polynomial $\sigma'_{ij}(0, t)$ is less than the sum of degrees of degeneracy of the eigenvalues γ_i and γ_j.

To calculate $\sigma'_{ij}(0, t)$ explicitly and to integrate the second term of the right-hand side of (63), it is convenient to note the following formula on the nilpotent matrix $\mathbf{N}_j$:

$$\exp(t\mathbf{N}_j) = \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n_j-1}/(n_j-1)! \\ & 1 & t & & t^{n_j-2}/(n_j-2)! \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix} \tag{67}$$

Now let γ_1 be the (maximum) eigenvalue whose real part $\mathrm{Re}\,\gamma_1$ is the maximum among all the eigenvalues $\{\gamma_j\}$. Then, the above linear approximation breaks down around the time t_1 which satisfies

$$|\epsilon\sigma'_{11}(t_1)| \sim \Delta, \qquad \text{or} \qquad |\sigma'_{11}(t_1)| \sim \Delta\epsilon^{-1} \tag{68}$$

or which is given by

$$t_1 \sim (2\,\mathrm{Re}\,\gamma_1)^{-1}\log(\Delta/\epsilon) \tag{69}$$

from Theorem 1. That is, the variance corresponding to the mode of the "maximum eigenvalue" γ_1 is seen to be anomalously enhanced around the time region (69), namely to be of order ϵ^{-1}, while the normal variance is of order unity. Then, the fluctuation enhancement factor R is given by $R \sim \epsilon^{-1}$.

Next, we discuss the time region for the anomalous enhancement of variances corresponding to other modes. From the above linear theory, they become anomalously enhanced at different time regions

$$t_{ij} \sim [2\,\mathrm{Re}(\gamma_i + \gamma_j)]^{-1} \log(\Delta/\epsilon) \tag{70}$$

if $\mathrm{Re}(\gamma_i + \gamma_j) > 0$. Here it should be remarked that the nonlinear effect neglected in the above linear approximation enhances anomalous fluctuations of other unstable modes in the same time region as (69), if there exists mode coupling among them. If they are separated into certain independent subspaces, then we have to discuss, of course, the anomalous enhancement of fluctuations in each subspace, by using each maximum eigenvalue in each subspace.

To see the nonlinear effect, we have to include other nonlinear terms into the expression of the probability distribution (54):

$$P(\mathbf{x}, t) \sim \exp\left[-(\mathbf{x} - \mathbf{x}_0)\frac{1}{2\epsilon\boldsymbol{\sigma}(t)}(\mathbf{x} - \mathbf{x}_0) + \text{higher terms}\right] \tag{71}$$

Correspondingly, the fluctuation matrix can be expanded as

$$\langle(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)\rangle = \epsilon\boldsymbol{\sigma}_1(t) + \epsilon^2\boldsymbol{\sigma}_2(t) + \epsilon^3\boldsymbol{\sigma}_3(t) + \cdots \tag{72}$$

with $\boldsymbol{\sigma}_1(t) \equiv \boldsymbol{\sigma}(t)$. It is easy to obtain formally the temporal evolution equations of the coefficients $\boldsymbol{\sigma}_n(t)$ in (72), on the basis of the following equation[1] for the average of an arbitrary quantity $Q(x)$:

$$\epsilon\frac{d}{dt}\langle Q(\mathbf{x})\rangle + \left\langle \mathscr{H}^*\left(\mathbf{x}, \epsilon\frac{\partial}{\partial\mathbf{x}}\right)Q(\mathbf{x})\right\rangle = 0 \tag{73}$$

where $\mathscr{H}^*$ is the adjoint operator of $\mathscr{H}$, defined by

$$\mathscr{H}^*(\mathbf{x}, \mathbf{p}) \equiv -\sum_{n=1}^{\infty}\frac{1}{n!}c_n(\mathbf{x})\mathbf{p}^n \tag{74}$$

That is, we obtain

$$\sum_{n=1}^{\infty}\epsilon^{n+1}\frac{d}{dt}\boldsymbol{\sigma}_n(t) + \left\langle \mathscr{H}^*\left(\mathbf{x}, \epsilon\frac{\partial}{\partial\mathbf{x}}\right)(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)\right\rangle = 0 \tag{75}$$

together with equations of motion in higher moments appearing in the second term of (75).

It is found from Eqs. (75), etc., that fluctuations corresponding to other unstable modes than the γ_1 mode become enhanced anomalously at the same time region as (69). More detailed arguments on these nonlinear effects or mode coupling will be given in Section 4 in the extensive region.

Next, we discuss the second, nonlinear or non-Gaussian regime. As shown in the above paragraph, the nonlinear regime starts at the time region

(69). The fluctuating part $\mathbf{z}(t)$ becomes of order unity in this nonlinear regime, and it satisfies the scaling property (46). Thus, the following inequality holds:

$$\left| \epsilon \frac{\partial^2}{\partial \mathbf{x}\, \partial \mathbf{x}} \mathbf{c}_2(\mathbf{x}) P \right| \ll \left| \frac{\partial}{\partial \mathbf{x}} \mathbf{c}_1(\mathbf{x}) P \right| \tag{76}$$

where the left-hand side is of order ϵ in the second regime. The nth term of the second expression in (40) is of order ϵ^n in the second, nonlinear regime. Thus, the Kramers–Moyal equation (40) can be reduced asymptotically to the following *drift equation*:

$$\frac{\partial}{\partial t} P(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}} c_1(\mathbf{x}) P(\mathbf{x}, t) = 0 \tag{77}$$

which corresponds to (16) for a single macrovariable. The characteristic equations of (77) are

$$(d/dt)\mathbf{x} = \mathbf{c}_1(\mathbf{x}) \tag{78}$$

and

$$(d/dt)P + P \operatorname{div} \mathbf{c}_1(\mathbf{x}) = 0 \tag{79}$$

Now let us assume that the solution of (78) with the initial condition $\mathbf{x} = \mathbf{y}_0$ at $t = t_0$ is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{y}_0, t) \tag{80}$$

and that the inverse solution of (80) is given by

$$\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x}, t) \tag{81}$$

Then, the solution of the drift equation is formally expressed by

$$\begin{aligned} P^{\mathrm{II}}(\mathbf{x}, t) &= \exp\left[-\int_{t_0}^{t} \operatorname{div} \mathbf{c}_1(\mathbf{x}(\mathbf{y}_0, s))\, ds \right]_{y_0 = y_0(\mathbf{x}, t)} P(\mathbf{y}_0(\mathbf{x}, t), t_0) \\ &= \text{Jacobian}\left(\frac{\partial \mathbf{y}_0}{\partial \mathbf{x}} \right)_{\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x}, t)} P(\mathbf{y}_0(\mathbf{x}, t), t_0) \end{aligned} \tag{82}$$

Therefore, the probability is conserved under the temporal evolution (77) or (78). These situations are shown schematically in Fig. 5.

Thus, the fluctuation in the second regime is non-Gaussian, but it is probabilistic or stochastic in the sense that the stochastic nature comes from the probability distribution in the initial regime and that each representative motion is deterministic. Namely, a random force can be neglected asymptotically in the second regime. This may be regarded as giving partially a conceptual foundation to the eddy turbulence theory of Tatsumi *et al.*,[18] who have studied the energy spectrum of the eddy turbulence in the (deterministic) Navier–Stokes equation by imposing an appropriate probability distribution on the initial condition.

The above arguments can be summarized in the following.

Theorem 2 (Fluctuation-Enhancement Theorem). A large enhancement of fluctuation occurs around the time region

$$t_m \sim (2 \operatorname{Re} \gamma_1)^{-1} \log(1/\epsilon) \tag{83}$$

where γ_1 is the "maximum" eigenvalue of the regression matrix $\mathbf{K}_0$. The enhancement factor R is given by

$$R \sim 1/\epsilon \tag{84}$$

when the initial system is located at (or very close to) the unstable point with a variance of order ϵ.

In the *final regime*, the fluctuation is again normal as shown in (46) and consequently the temporal evolution of the system is asymptotically described by the linear Fokker–Planck equation (51), where

$$\boldsymbol{\gamma}_e = \mathbf{c}_1'(\mathbf{x}_e) \equiv \frac{\partial}{\partial \mathbf{x}_e} \mathbf{c}_1(\mathbf{x}_e), \qquad \mathbf{D}_e = (\mathbf{c}_2^{ij}(\mathbf{x}_e)) \tag{85}$$

with a stable equilibrium point $\mathbf{x}_e$, if it exists. The solution of (51) with the initial condition

$$P_0(\mathbf{x}) = P^{\mathrm{II}}(\mathbf{x}, t_2) \tag{86}$$

at a time t_2 in the boundary region between the second and final regimes is expressed by

$$P_{\mathrm{fin}}(\mathbf{x}, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \frac{1}{[\det \boldsymbol{\sigma}(t)]^{1/2}} \int_{-\infty}^{\infty} P_0(\mathbf{y}) \times \exp\left\{-[\mathbf{x} - \mathbf{y}(t)] \frac{1}{2\epsilon\boldsymbol{\sigma}_f(t)} [\mathbf{x} - \mathbf{y}(t)]\right\} d^n\mathbf{y} \tag{87}$$

where

$$\mathbf{y}(t) = \{\exp[\boldsymbol{\gamma}_e(t - t_2)]\}(\mathbf{y} - \mathbf{x}_e) + \mathbf{x}_e$$

and

$$(d/dt)\boldsymbol{\sigma}_f(t) = \boldsymbol{\gamma}_e\boldsymbol{\sigma}_f(t) + \boldsymbol{\sigma}_f(t)\tilde{\boldsymbol{\gamma}}_e + \mathbf{D}_e \tag{88}$$

with $\boldsymbol{\sigma}_f(t_2) = 0$. The solution of (88) is given by

$$\boldsymbol{\sigma}_f(t) = \int_{t_2}^{t} \exp[\boldsymbol{\gamma}_e(t - t')]\, \mathbf{D}_e \exp[\tilde{\boldsymbol{\gamma}}_e(t - t')]\, dt' \tag{89}$$

Thus we obtain

$$\boldsymbol{\sigma}_f(\infty) = \int_0^{\infty} \exp(\boldsymbol{\gamma}_e t)\, \mathbf{D}_e \exp(\tilde{\boldsymbol{\gamma}}_e t)\, dt \tag{90}$$

Therefore, $P_{\text{fin}}(\mathbf{x}, t)$ approaches asymptotically the correct stable equilibrium solution for $t \to \infty$. The above solution (87) is seen to be qualitatively independent of the choice of t_2. The above connection procedure can be extended to the case where there exist several stable equilibrium points, by dividing appropriately the n-dimensional x space into subregions each of which contains only one stable equilibrium point inside, as was discussed in Section 2 for a single macrovariable. When there does not exist a definite stable equilibrium point $\mathbf{x}_e$ as in the case of a limit cycle, we need a more complicated treatment, which will be discussed in the future.

4. ANOMALOUS FLUCTUATION EFFECT IN THE EXTENSIVE REGION

In the preceding sections, we have discussed fluctuation and relaxation in the unstable region $\delta^2 \lesssim \epsilon$ as shown in Fig. 3. Here, δ denotes a typical magnitude of the deviation of the initial system from the unstable point (say $\delta \sim |\mathbf{y}_0 - \mathbf{x}_0|$). Now, in this section we study the anomalous fluctuation effect[2,10] in the extensive region $\epsilon \ll \delta^2$. Since the fluctuating part $\mathbf{z}(t)$ of the intensive macrovariable $\mathbf{x}$ is normal, namely Gaussian in this extensive region, we put

$$\mathbf{x}(t) = \mathbf{y}(t) + \xi(t)\Omega^{-1/2} \tag{91}$$

in (40), following van Kampen,[11] and expand (40) with respect to the smallness parameter ϵ. Alternatively, following Kubo *et al.*,[10] we may also apply the extensivity Ansatz[10,12,15] to (40). Anyway, the solution takes the following form[10,14]:

$$P(\mathbf{x}, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \frac{1}{[\det \boldsymbol{\sigma}(t)]^{1/2}} \exp\left\{-[\mathbf{x} - \mathbf{y}(t)] \frac{1}{2\epsilon\boldsymbol{\sigma}(t)} [\mathbf{x} - \mathbf{y}(t)]\right\} \tag{92}$$

asymptotically for a small ϵ, where

$$(d/dt)\mathbf{y}(t) = \mathbf{c}_1(\mathbf{y}(t)) \tag{93}$$

and the variance $\boldsymbol{\sigma}(t)$ is determined by

$$(d/dt)\boldsymbol{\sigma}(t) = \mathbf{K}(\mathbf{y})\boldsymbol{\sigma}(t) + \boldsymbol{\sigma}(t)\tilde{\mathbf{K}}(\mathbf{y}) + \mathbf{D}(\mathbf{y}) \tag{94}$$

with

$$\mathbf{K}(\mathbf{y}) = \left(\frac{\partial}{\partial y_j} c_1{}^i(\mathbf{y})\right), \qquad \mathbf{D}(\mathbf{y}) = \mathbf{c}_2(\mathbf{y}) \tag{95}$$

In order to study the anomalous behavior of $\mathbf{y}(t)$ and $\boldsymbol{\sigma}(t)$ near the instability point ($\delta \ll 1$), we first linearize Eqs. (93) and (94) as

$$\frac{d}{dt}\,\delta\mathbf{y}(t) = \mathbf{K}_0\,\delta\mathbf{y}(t), \qquad \frac{d}{dt}\,\boldsymbol{\sigma}(t) = \mathbf{K}_0\boldsymbol{\sigma} + \boldsymbol{\sigma}\tilde{\mathbf{K}}_0 + \mathbf{D}_0 \tag{96}$$

near the unstable point $\mathbf{x}_0$, where $\mathbf{K}_0$ and $\mathbf{D}_0$ are given by (53), and $\delta\mathbf{y}(t) = \mathbf{y}(t) - \mathbf{y}_0$. Similar to the case for a single macrovariable in Section 2, the solution $\delta\mathbf{y}(t)$ is expressed as

$$\delta\mathbf{y}(t) = [\exp(\mathbf{K}_0 t)]\, \delta\mathbf{y}_0 \equiv [\exp(\mathbf{K}_0 t)]\, \boldsymbol{\delta} \tag{97}$$

The second equation of (96) is the same as (55) in Section 3. Thus, the analyses of $\exp(\mathbf{K}_0 t)$ and $\boldsymbol{\sigma}(t)$ in Section 3 can be immediately applied to the present problem. By the use of the regular transformation $\mathbf{P}$ in (64), we obtain

$$(\delta\mathbf{y}(t))' \equiv \mathbf{P}\, \delta\mathbf{y}(t) = [\exp(\mathbf{K}_0{}' t)]\, \boldsymbol{\delta}' = [\exp(\mathbf{S}' t) \exp(\mathbf{N}' t)]\, \boldsymbol{\delta}' \tag{98}$$

Therefore the jth element (or subvector) of (98) is written as

$$(\delta\mathbf{y}(t))_j{}' = e^{\gamma_j t}(e^{t\mathbf{N}_j}\, \boldsymbol{\delta}_j{}') \equiv e^{\gamma_j t}\, \boldsymbol{\delta}_j{}'(t) \tag{99}$$

Here $\boldsymbol{\delta}_j{}'(t)$ is a polynomial in t of order less than n_j (where n_j denotes the degeneracy of the eigenvalue γ_j). As before, let γ_1 be the (maximum) eigenvalue whose real part Re γ_1 is the maximum. Then the above linear approximation is valid at most only up to the time t_1 which satisfies

$$(\delta y(t_1))_1{}' \equiv \delta \exp(\mathrm{Re}\, t_1) = \Delta(\mathrm{const} \ll 1) \tag{100}$$

where $\delta = \delta_1{}'(t_1) \sim \delta_1{}'$ (a deviation of the "maximum" γ_1 mode at the initial time). That is, t_1 is given by

$$t_1 = (\mathrm{Re}\, \gamma_1)^{-1} \log(\Delta/\delta) \tag{101}$$

which corresponds to (37) for a single macrovariable. Next, the variance $\sigma_{11}'(t)$ corresponding to the γ_1 mode is given by

$$\sigma_{11}'(t) = \hat{\sigma}_{11}(0, t) \exp(2\gamma_1 t) + \mathrm{const} \tag{102}$$

from (66) in Section 3. This variance becomes of the order

$$\sigma_m \equiv \hat{\sigma}_{11}(0, t_1) \exp(2\, \mathrm{Re}\, \gamma_1 t) = \hat{\sigma}_{11}(0, t_1)(\Delta/\delta)^2 \sim 1/\delta^2 \tag{103}$$

Namely, the variance of the maximum mode is seen to be enhanced more and more, in proportion to the inverse square of the deviation δ of the initial system from the unstable point, as the time increases beyond the region (101). However, the saturation of the fluctuation comes from the nonlinear effect neglected in the above linear approximation. This will be discussed later. It should be remarked here that the maximum enhancement in the extensive region should be much less than that in the unstable region, that is,

$$\delta^{-2} \ll \epsilon^{-1} \qquad \text{or} \qquad \epsilon \ll \delta^2 \tag{104}$$

because the order of instability here is much smaller than that in the unstable region. The inequality (104) is nothing but the criterion of the extensive region in Fig. 3.

Next, we discuss the nonlinear effect or mode-coupling effect in order to investigate fluctuations of other modes than the dominant γ_1 mode, and to study the saturation effect.

For this purpose it is convenient to introduce the following extended vector representations for the matrices $\boldsymbol{\sigma}(t)$ and $\mathbf{D}(t)$:

$$\begin{aligned} \widetilde{(\vec{\sigma})} &= (\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}, \sigma_{22}, \sigma_{23}, \ldots, \sigma_{2n}, \sigma_{33}, \ldots, \sigma_{nn}) \\ \widetilde{(\vec{D})} &= (D_{11}, D_{12}, \ldots, D_{1n}, D_{22}, D_{23}, \ldots, D_{2n}, D_{33}, \ldots, D_{nn}) \end{aligned} \tag{105}$$

For later convenience, the procedure to make the above extended vector $\vec{\sigma}$ from the original matrix $\boldsymbol{\sigma}$ may be written as

$$\vec{\sigma} = \mathscr{F}(\boldsymbol{\sigma}) \tag{106}$$

Then, the temporal evolution (94) of the variance $\boldsymbol{\sigma}(t)$ is rewritten as

$$\frac{d}{dt}\vec{\sigma}(t) = S(\mathbf{y}(t))\vec{\sigma}(t) + \vec{D}(t) \tag{107}$$

where $S(\mathbf{y}(t))$ is an extended matrix to project $\boldsymbol{\sigma}(t)$ to $(\mathbf{K}\boldsymbol{\sigma} + \boldsymbol{\sigma}\tilde{\mathbf{K}})$, namely

$$S(\mathbf{y}(t))\vec{\sigma} \equiv \mathscr{F}(\mathbf{K}\boldsymbol{\sigma} + \boldsymbol{\sigma}\tilde{\mathbf{K}}) \tag{108}$$

The dimensionality of $S(\mathbf{y}(t))$ is $n(n + 1)/2$. By expanding (107) around the unstable point, we obtain the linear approximation corresponding to (96) in the form

$$\frac{d}{dt}\vec{\sigma}_l = S_0\vec{\sigma}_l + \vec{D}_0; \qquad S_0 = S(\mathbf{y}_0), \qquad \text{etc.} \tag{109}$$

The solution of this equation is given by

$$\vec{\sigma}_l(t) = e^{S_0 t}(\vec{\sigma}_l(0) - \vec{\sigma}_0) + \vec{\sigma}_0 \tag{110}$$

if there exists a solution $\vec{\sigma}_0$ of the equation $S_0\vec{\sigma}_0 + \vec{D}_0 = 0$, i.e., if S_0 is regular. This is an alternative expression of (57). Namely, we have $\sigma_l(t) = \mathscr{F}(\boldsymbol{\sigma}_l(t))$.

For simplicity we consider here the case in which $\mathbf{K}_0$ is semisimple, i.e., diagonalizable. (The argument for the general case is quite similar, but more complicated expressions appear.) Then, we have

$$\mathbf{K}_0' = \mathbf{P}\mathbf{K}_0\mathbf{P}^{-1} = \begin{pmatrix} \gamma_1 & & & 0 \\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_n \end{pmatrix} \tag{111}$$

The corresponding variance $\sigma_l'(t)$ shows the following time dependence:

$$\vec{\sigma}_l'(t) = \mathscr{F}(\sigma_l'(t)) = \mathscr{F}(\mathbf{P}\sigma_l(t)\tilde{\mathbf{P}}) = \begin{pmatrix} \sigma_{11}'e^{2\gamma_1 t} \\ \sigma_{12}'e^{(\gamma_1+\gamma_2)t} \\ \vdots \\ \sigma_{ij}'e^{(\gamma_i+\gamma_j)t} \\ \vdots \\ \sigma_{nn}'e^{2\gamma_n t} \end{pmatrix} + \vec{\sigma}_0' \qquad (112)$$

That is, the eigenvalues of S_0 are given by $2\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_i + \gamma_j, \ldots, 2\gamma_n$, and S_0' transformed from S_0 in terms of $\mathbf{P}$ is diagonal. However, $S'(\mathbf{y}(t))$ is not necessarily diagonal, because of nonlinear coupling among modes. Then, we put

$$S'(\mathbf{y}(t)) = S_0' + \delta S'(\mathbf{y}(t)); \qquad S_0' \equiv S'(\mathbf{y}_0) \qquad (113)$$

If $S'(\mathbf{y}(t))$ is separated into some independent subspaces, then the following argument can be made in each subspace. Thus, from the beginning, we confine our arguments into an irreducible matrix $S'(\mathbf{y}(t))$ or an intrinsically coupled system. In such an irreducible space, we expand $\delta S'(\mathbf{y}(t))$ with respect to $\delta\mathbf{y}'(t) \equiv \mathbf{P}\,\delta\mathbf{y}(t) \equiv \mathbf{P}(\mathbf{y}(t) - \mathbf{y}_0)$ as follows:

$$\delta S'(\mathbf{y}(t)) = \sum L_i'\,\delta y_i'(t) + \sum_{i,j} L_{ij}'\,\delta y_i'(t)\,\delta y_j'(t) + \cdots \qquad (114)$$

where all $L_i', L_{ij}', \ldots$ are matrices of the same order as $\delta S'(\mathbf{y}(t))$. Thus, the temporal evolutions of $\delta\mathbf{y}'(t)$ and $\boldsymbol{\sigma}'(t)$ are described by the following nonlinear coupled equations:

$$(d/dt)\,\delta\mathbf{y}'(t) = \mathbf{K}_0'\,\delta\mathbf{y}'(t) + \delta\mathbf{y}'(t)\mathbf{K}_1'\,\delta\mathbf{y}'(t) + \cdots \qquad (115)$$

$$(d/dt)\vec{\sigma}'(t) = \left(S_0' + \sum_i L_i'\,\delta y_i'(t) + \cdots\right)\vec{\sigma}'(t) + \vec{D}' \qquad (116)$$

with appropriate coefficients $\mathbf{K}_1'$ and L_i'. From the irreducibility of $\delta S'(\mathbf{y}(t))$, there exists at least one nonvanishing mode coupling. For example, the cross variance $\sigma_{12}'(t)$ corresponding to the γ_1 and γ_2 modes satisfies the following equation:

$$(d/dt)\sigma_{12}'(t) = (\gamma_1 + \gamma_2)\sigma_{12}'(t) + \alpha_{21}\,\delta y_1'(t)\sigma_{11}'(t) + \cdots \qquad (117)$$

with an appropriate nonvanishing constant α_{21}. The solution of (117) is given by

$$\sigma_{12}'(t) = \{\exp[(\gamma_1 + \gamma_2)t]\}\left\{\int_0^t \alpha_{21}\,\delta y_1'(t')\sigma_{11}'(t')\exp[-(\gamma_1 + \gamma_2)t']\,dt' + \cdots\right\}$$

$$= \frac{\alpha_{21}\sigma_{11}'(0)}{2\gamma_1 - \gamma_2}\,\delta\exp(3\gamma_1 t) + \text{const} \times \exp[(\gamma_1 + \gamma_2)t] + \cdots \qquad (118)$$

where we have put $\delta y_1'(t) \simeq \delta e^{\gamma_1 t}$. Since $\delta \exp(\mathrm{Re}\,\gamma_1 t_m) \simeq \Delta$ (i.e., of order unity) in the second anomalous fluctuation regime, we obtain

$$\sigma_{12}'(t_m) \simeq \frac{\alpha_{21}\sigma_{11}'(0)}{2\gamma_1 - \gamma_2} \exp(2\gamma_1 t_m) \simeq \frac{1}{\delta^2} \tag{119}$$

Thus, the cross variance shows an anomalous fluctuation ($\propto \delta^{-2}$) similar to $\sigma_{11}'(t)$ for the time region $t_m \sim (\mathrm{Re}\,\gamma_1)^{-1} \log(1/\delta)$. The above arguments may be easily extended to any other general mode. Then, we obtain the following theorem.

Theorem 3 (Anomalous Fluctuation Theorem). The variance $\boldsymbol{\sigma}(t)$ shows an anomalous enhancement

$$\sigma_m \sim \delta^{-2} \qquad \text{at} \qquad t_m \sim (\mathrm{Re}\,\gamma_1)^{-1} \log(1/\delta) \tag{120}$$

for coupled unstable modes.

This is a generalization of the anomalous fluctuation theorem proven in a previous paper[2] for a single macrovariable.

We may also discuss the nonlinear effect of the most probable path $\mathbf{y}(t)$, and it is seen that $\mathbf{y}(t)$ becomes of order unity for the time region $t_m \sim (\mathrm{Re}\,\gamma_1)^{-1} \log(1/\delta)$. Clearly, the saturation effect comes from the nonlinear coupling among modes. For the specific case of two macrovariables, a more detailed study has been made by Saito and Kubo[19] in the kinetic Bethe lattice, in which two macrovariables of long-range and short-range order parameters show similar anomalous fluctuations ($\sim \delta^{-2}$) for the time region $t_m \sim (\mathrm{Re}\,\gamma_1)^{-1} \log(1/\delta)$ near the instability point ($\delta \ll 1$).

5. SUMMARY AND DISCUSSION

The scaling theory of transient phenomena for a single mode near the instability point has been extended to multimode systems. The time region has been divided into three regimes, namely (i) the initial, linear, Gaussian regime, (ii) the second, nonlinear, non-Gaussian, anomalous fluctuation, or drift regime, and (iii) the final, Gaussian regime. It has been proven that there occurs a large enhancement of fluctuation of relative order ϵ^{-1} for the time region (83), when the initial system is just at (or close to) the unstable point (*fluctuation-enhancement theorem*). In the extensive region ($\epsilon \ll \delta^2$), the *anomalous-fluctuation theorem* has been established: The variance $\boldsymbol{\sigma}(t)$ shows an anomalous enhancement proportional to δ^{-2} for the time region $t_m \sim (\mathrm{Re}\,\gamma_1)^{-1} \log(1/\delta)$ if the initial system deviates by δ from the instability point.

Our present argument can be extended to more refined treatments to divide the time region into more than three regimes or an infinite number of regimes. Although we have discussed mainly a finite number of multi-macrovariables in the present paper, the fundamental idea will be useful even for an infinite number of macrovariables, as suggested in Eqs. (44)–(48). For example, the temporal evolution of the system in the second regime is described by the drift equation

$$\frac{\partial}{\partial t} P(\{x_k\}) + \sum_k \frac{\partial}{\partial x_k} c_1(\{x_k\}) P(\{x_k\}, t) = 0 \tag{121}$$

although for practical calculation it still needs an approximation such as by a perturbation calculation. Such an extension has been tried by Kawasaki[20] in the time-dependent Ginzburg–Landau equation. An application to a chemical reaction of two components is discussed in Appendix C.

APPENDIX A. RELAXATION IN THE LASER MODEL

The scaling solution for the laser model with $c_1(x) = \gamma x(1 - x^2)$ and $c_2 = 2$ is given by[1,4]

$$P_{sc}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} \exp\left[-\frac{x^2}{2\tau(1 - x^2)} - \frac{3}{2}\log(1 - x^2) \right] \tag{A.1}$$

with $\tau = \epsilon(\sigma_0 + \sigma_1)\exp(2\gamma t)$ and $\sigma_1 = \gamma^{-1}$. Consequently, the solution in the final regime is expressed by (31a) or (31b) with (A.1). Here, we put $\tau(t_2) = 1$, which corresponds to the time when double peaks appear appreciably near $x_e = \pm 1$. (Note that the transition time from a single peak to double peaks is given by $\tau_0 = 1/3$ in the scaling theory.[1,4]) Thus, an explicit result after integrating (31a) for (A.1) is shown in Fig. 6. A new point is that the proba-

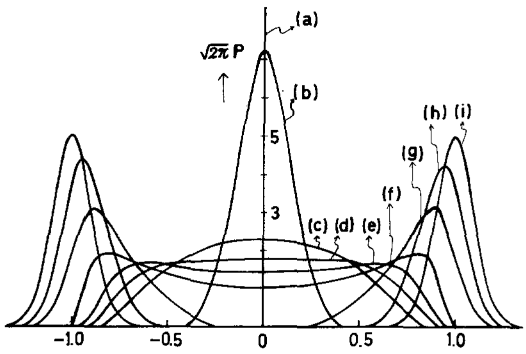

Fig. 6. Change of the distribution function: (a) $P_0 = \delta(x)$ at $t = 0$; (b) $\tau = 0.02$; (c) $\tau = 0.2$; (d) $\tau = \tau_0 = 1/3$; (e) $\tau = 0.5$; (f) $\tau = 1$; (g) $\tau = 2$; (h) $\tau = 4$; and (i) P_{eq} at $t = \infty$, where $\sigma_f = 1$, and $\tau = 0.01$.

bility distribution function in the final regime has been calculated and that it is spread outside the equilibrium points $x_e = \pm 1$, as it should be.

APPENDIX B. AN ALTERNATIVE FORMULATION OF THE SCALING THEORY

The scaling results (23) and (25) suggest an alternative formulation of the scaling theory, which assumes the Ansatz (39); namely

$$P(x, t) \cong C \exp[(1/\epsilon)\varphi_0(x, t) + \varphi_1(x, t) + \epsilon\varphi_2(x, t) + \cdots] \tag{B.1}$$

Here, the second term $\varphi_1(x, t)$ in (39) or (B.1) may be of the same order of magnitude as the first extensive part $\epsilon^{-1}\varphi_0(x, t)$ in the second regime. However, we assume here that the temporal evolutions are determined by equating[(10,12),4] formally the power series expansions of both sides of (13) after substituting (B.1) into (13), to yield the following results:

$$\frac{\partial\varphi_0}{\partial t} = -\int w(x, r)\left[1 - \exp\left(-r\frac{\partial\varphi_0}{\partial x}\right)\right] dr \tag{B.2}$$

$$\frac{\partial\varphi_1}{\partial t} = \int \left[w(x, r)\left(\frac{r^2}{2}\frac{\partial^2\varphi_0}{\partial x^2} - r\frac{\partial\varphi_1}{\partial x}\right) - r\frac{\partial w}{\partial x}\right] \exp\left(-r\frac{\partial\varphi_0}{\partial x}\right) dr \tag{B.3}$$

and

$$\begin{aligned}\frac{\partial\varphi_2}{\partial t} = \int &\left[\frac{r^2}{2}\frac{\partial^2 w}{\partial x^2} + r^2\frac{\partial w}{\partial x}\frac{\partial\varphi_1}{\partial x} - \frac{r^3}{2}\left(\frac{\partial w}{\partial x} + w\frac{\partial\varphi_1}{\partial x}\right)\frac{\partial^2\varphi_0}{\partial x^2}\right.\\ &\left. + w\left\{\frac{r^4}{8}\left(\frac{\partial^2\varphi_0}{\partial x_2}\right)^2 + \frac{r^2}{2}\left(\frac{\partial^2\varphi_1}{\partial x^2}\right)^2 - \frac{r^3}{6}\left(\frac{\partial^3\varphi_0}{\partial x^3}\right) + \frac{r^2}{2}\left(\frac{\partial\varphi_1}{\partial x}\right)^2 - r\left(\frac{\partial\varphi_2}{\partial x}\right)\right\}\right]\\ &\times \exp\left(-r\frac{\partial\varphi_0}{\partial x}\right) dr\end{aligned} \tag{B.4}$$

In greater generality we have[5]

$$\frac{\partial\varphi_n}{\partial t} = \int f_n(x, r, t)\, dr; \qquad f_n = \sum_{k=0}^{n}\frac{(-r)^{n-k}}{(n-k)!}g_k w^{(n-k)} \tag{B.5}$$

where $w^{(n)} = \partial^n w/\partial x^n$, $g_0 = 1$, and

$$\begin{aligned} g_n &= \sum_{\substack{m_1+2m_2+\ldots=n\\(m_1\geq 0, m_2\geq 0,\ldots)}} \left[\exp - r\left(\frac{\partial\varphi_0}{\partial x}\right)\right]\frac{s_1^{m_1}s_2^{m_2}\cdots}{m_1!\,m_2!\cdots}\\ s_k &= \sum_{m=1}^{k+1}\frac{(-r)^m}{m!}\varphi_{k+1-m}^{(m)}(x, t)\end{aligned} \tag{B.6}$$

[4] Equations (9.7) and (9.8) in Ref. 12 should be corrected to read (B.5) and (B.6), respectively.

[5] See footnote 4.

As was shown in Ref. 12, Eq. (B.5) takes the form

$$\frac{\partial \varphi_n(x, t)}{\partial t} + c_1(x) \frac{\partial \varphi_n}{\partial x} = R_n(x, \varphi_0, \ldots, \varphi_{n-1}) \tag{B.7}$$

for $n \geqslant 1$. These can be solved formally for $n \geqslant 1$ by the Lagrange method. From (B.2), $\varphi_0(x, t)$ satisfies the equation

$$\frac{\partial \varphi_0}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} c_n(x) \left(\frac{\partial \varphi_0}{\partial x}\right)^n \tag{B.8}$$

The solution of this equation with the unstable initial condition[1-4] $\varphi_0(x) = -x^2/(2\sigma_0)$ for $c_1(0) = 0$ and $\gamma \equiv c_1'(0) > 0$ can be easily shown to take the following asymptotic form:

$$\varphi_0(x, t) \sim e^{-2\gamma t} \varphi_0(x) \tag{B.9}$$

for large t. Namely, $\varphi_0(x, t)$ goes to zero as t increases. Therefore, (B.8) can be reduced asymptotically to the following simple linear equation:

$$\partial \varphi_0 / \partial t = -c_1(x)\, \partial \varphi_0 / \partial x \tag{B.10}$$

in the second nonlinear regime (i.e., $e^{-2\gamma t} \sim \epsilon \ll 1$), because $(\partial \varphi_0 / \partial x)^n \sim \exp(-2n\gamma t) \ll \exp(-2\gamma t)$ for $n \geqslant 2$. Similarly, (B.3) can be reduced to

$$\frac{\partial \varphi_1(x, t)}{\partial t} = -c_1(x) \frac{\partial \varphi_1}{\partial x} - \frac{d}{dx} c_1(x) \tag{B.11}$$

in the second regime. Clearly, the asymptotic solutions of (B.10) and (B.11) with appropriate initial conditions constitute the scaling results (23) and (25).

All the other terms $\varphi_n(x, t)$ are shown to be of order unity in the second regime, except for a linearly divergent term $[\propto(-\gamma t)]$, which comes from the normalization of the probability distribution function. Thus, the terms $\{\epsilon^n \varphi_n(x, t)\}$ for $n \geqslant 2$ can be neglected in the second regime. That is, the second term $\varphi_1(x, t)$ is essential in the second scaling regime.

A justification of the above derivation is given by the fact that the results thus obtained agree with those obtained by the rigorous scaling theory. The present formulation is useful in treating both unstable and extensive regions in a unified way.

APPENDIX C. RELAXATION AND FLUCTUATION OF CHEMICAL REACTION IN BRUXELLATOR (PLN MODEL)

As an application of the general theory given in the text, we discuss here the fluctuation and relaxation of the Prigogine–Lefever–Nicolis (PLN) model,[21,22] which will be described by the following nonlinear Fokker–Planck equation[22]:

$$\frac{\partial}{\partial t}P = \left[-\frac{\partial}{\partial \mathbf{x}}\mathbf{c}_1(\mathbf{x}) + \frac{\epsilon}{2}\frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{D}\cdot\frac{\partial}{\partial \mathbf{x}}\right]P \tag{C.1}$$

where the first moment (vector) $\mathbf{c}_1$ and diffusion matrix $\mathbf{D}$ are given by

$$\mathbf{c}_1 = \begin{pmatrix} x^2y - bx - a - x \\ bx - x^2y \end{pmatrix}; \qquad a > 0, \quad b > 0 \tag{C.2}$$

and

$$\mathbf{D} = \begin{pmatrix} x^2y + a + x + bx, & -x^2y - bx \\ -x^2y - bx, & x^2y + bx \end{pmatrix} \tag{C.3}$$

with the same notations as in Refs. 21 and 22. Namely, x and y denote the concentrations of two kinds of chemical species, and a and b are constants related to reaction rates. The unstable equilibrium point is given by

$$\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}; \qquad x_0 = a \quad y_0 = b/a \tag{C.4}$$

From the general theory in Section 3, we first linearize (C.1) as (52) with

$$\mathbf{K}_0 = \mathbf{c}_1'(\mathbf{x}_0) = \begin{pmatrix} b - 1, & a^2 \\ -b, & -a^2 \end{pmatrix} \tag{C.5}$$

and

$$\mathbf{D}_0 = \begin{pmatrix} 2ab + 2a, & -2ab \\ -2ab, & 2ab \end{pmatrix} \tag{C.6}$$

With these matrices, the solution $\delta\mathbf{y}(t) = \mathbf{y}(t) - \mathbf{x}_0$ of (96) is given by

$$\begin{aligned}\delta\mathbf{y}(t) &= [\exp(t\mathbf{K}_0)]\,\delta\mathbf{y}(0) \\ &= \exp\left(\frac{b - b_c}{2}t\right)\left[\cos(\omega t) - \left(\mathbf{K}_0 - \frac{b - b_c}{2}\right)\omega^{-1}\sin\omega t\right]\delta\mathbf{y}(0)\end{aligned} \tag{C.7}$$

where

$$\omega = \{a^2 - \tfrac{1}{4}(b - b_c)^2\}^{1/2}, \qquad b_c = a^2 + 1 \tag{C.8}$$

The solution $\boldsymbol{\sigma}_l(t)$ of (96) is also given by

$$\begin{aligned}\boldsymbol{\sigma}(t) &= [\exp(t\mathbf{K}_0)][\boldsymbol{\sigma}_l(0) - \sigma_0]\exp(t\tilde{\mathbf{K}}_0) + \boldsymbol{\sigma}_0 \\ &= \{\exp[(b - b_c)t]\widetilde{\boldsymbol{\sigma}_0}(t) + \boldsymbol{\sigma}_0\end{aligned} \tag{C.9}$$

where $\widetilde{\boldsymbol{\sigma}_0}(t)$ is given from (57) as

$$\begin{aligned}\widetilde{\boldsymbol{\sigma}_0}(t) &= \left(\cos\omega t + \frac{b - b_c}{2\omega}\sin\omega t - \frac{\mathbf{K}_0}{\omega}\sin\omega t\right) \\ &\quad \times [\boldsymbol{\sigma}_l(0) - \boldsymbol{\sigma}_0] \times \left(\cos\omega t + \frac{b - b_c}{2\omega}\sin\omega t - \frac{\tilde{\mathbf{K}}_0}{\omega}\sin\omega t\right)\end{aligned} \tag{C.10}$$

Thus, for the case that ω is real (i.e., $2a > b - b_c > 0$), $\delta\mathbf{y}(t)$ and $\boldsymbol{\sigma}_l(t)$ show the following asymptotic behavior:

$$\delta\mathbf{y}(t) \cong \exp[(b - b_c)t/2] \times (\text{finite vector}) \tag{C.11}$$

and

$$\boldsymbol{\sigma}_l(t) \cong \exp[(b - b_c)t] \times (\text{finite matrix}) \tag{C.12}$$

Therefore, the general arguments in Section 3 (see Theorem 2) lead to the following conclusion: When the initial system is located just at the unstable equilibrium point $\mathbf{x}_0$, the variance $\boldsymbol{\sigma}(t)$ is enhanced up to the order

$$\sigma_m \simeq \frac{1}{\epsilon} \qquad \text{at} \quad t_m \simeq \frac{1}{b - b_c} \log \frac{1}{\epsilon} \tag{C.13}$$

The new aspect of this result is that the maximum time t_m becomes larger and larger as the parameter b approaches the critical value b_c defined by (C.8). This is nothing but *the critical slowing down.* Thus, a large enhancement of fluctuation is expected to be observed experimentally in the future if the initial system is near the unstable equilibrium point (C.4). The time to observe this effect will be longer for b near b_c. In this chemical reaction of two components, there exist two kinds of modes whose amplification rates near the instability point $\mathbf{x}_0$ are

$$\gamma_{1,2} = \tfrac{1}{2}(b - b_c \pm \{[b - (a + 1)^2][b - (a - 1)^2]\}^{1/2}) \tag{C.14}$$

For the case that $2a > b - b_c > 0$ considered above, these two values are complex and they take the form

$$\gamma_{1,2} = \tfrac{1}{2}(b - b_c) \pm i\omega \tag{C.15}$$

This leads to the expression (C.10). The point $\mathbf{x}_0$ is a *spiral* unstable equilibrium point. On the other hand, for the noncritical case that $b > (a + 1)^2$ $(> b_c)$, both γ_1 and γ_2 are real, and $\mathbf{x}_0$ is a *nodal* point. The dominant rate constant is given by γ_1, and consequently, from the general arguments given in Section 3, the variance $\boldsymbol{\sigma}(t)$ is enhanced up to the order

$$\sigma_m \simeq \frac{1}{\epsilon} \qquad \text{at} \quad t_m \simeq \frac{1}{2\gamma_1} \log \frac{1}{\epsilon} \tag{C.16}$$

In the extensive region in which $\epsilon \ll \delta^2$ (where δ denotes the deviation of the initial system from the unstable point; $\delta \simeq |\mathbf{y}_0 - \mathbf{x}_0|$), the distribution function $P(\mathbf{x}, t)$ is expressed in the Gaussian form (92) asymptotically for a small ϵ with Eqs. (93), (94), (C.2), and (C.3) and with[22]

$$\mathbf{K}(\mathbf{y}) = \begin{pmatrix} 2xy - b - 1, & x^2 \\ b - 2xy, & -x^2 \end{pmatrix} \tag{C.17}$$

It is still difficult to solve analytically Eq. (93) for (C.2), because it is highly nonlinear. However, as was shown generally in Section 4, the anomalous fluctuation effect of this two-component system can be discussed by combining the behavior of the fluctuation in the initial linear regime and the nonlinear effect (which is usually taken into account perturbationally) in the second regime. Thus, we discuss first the solutions of Eqs. (93) and (94) with (C.5) and (C.6) instead (C.2) and (C.3). This corresponds to the linearization. Therefore, the solutions are given by (C.7) and (C.9). Namely,

$$|\delta \mathbf{y}(t)| \simeq \delta \exp\left[\left(\frac{b - b_c}{2}\right)t\right] \tag{C.18}$$

As was discussed in Section 4, the above linear approximation is valid at most only up to the time t_1 which satisfies

$$\delta \exp\left(\frac{b - b_c}{2} t_1\right) \simeq \Delta \tag{C.19}$$

in the region $2a > b - b_c > 0$. That is, t_1 is given by

$$t_1 \simeq \frac{2}{b - b_c} \log \frac{\Delta}{\delta} \tag{C.20}$$

Correspondingly, the variance becomes of the order

$$\sigma_m \simeq \exp(b - b_c)t_1 \simeq (\Delta/\delta)^2 \sim 1/\delta^2 \tag{C.21}$$

Namely, the variance of the maximum mode is enhanced proportionally to the inverse square of the deviation δ of the initial system form the unstable point.

These anomalous effects are expected to be observed experimentally in the future.

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